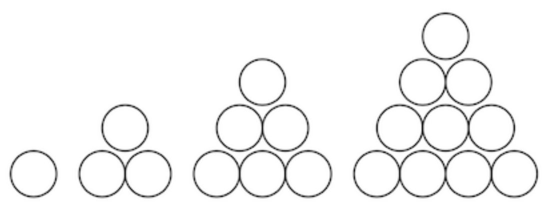
Grade 9/10 Math Circles

February 15

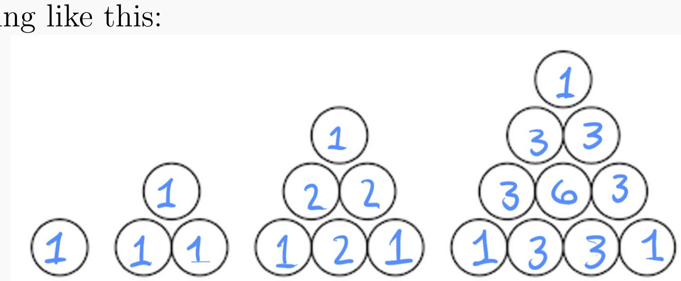
The Shape of You - Solution Set

Please put on your Real 3D™ glasses

1. Here's one possible way to keep track of the numbers in the 3D Pascal's Triangle:



(Write down the number of ways to get from the top to a particular intersection in each layer, always moving downwards, in the corresponding circles).

> *Solution*: Something like this:
>
> 
>

2. Here's something interesting I just noticed though: Is there a pattern to the *number of* circles in each layer above? Can you predict how many circles there will be in the next couple of layers, even without drawing all the circles?

> *Solution*: These numbers are so special that mathematicians gave them a name! They're called **triangular numbers** (guess why). The nth triangular number is equal to just the sum $1 + 2 + 3 + \ldots + n$. There's a famous story that a very young Carl Friedrich Gauss quickly calculated that the 100th triangular number is 5050 in his head. How? Imagine

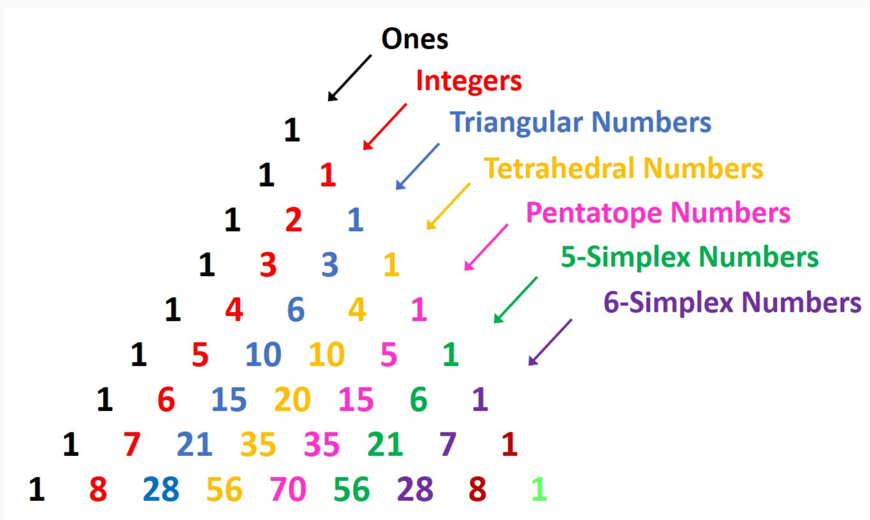
doubling the numbers that we're adding and lining them up so that they're easier to add:

$$\begin{array}{rrrrr} & 1\,+ & 2\,+ & \dots & +\,100 \\ + & 100\,+ & 99\,+ & \dots & +\,1 \\ \hline & 101\,+ & 101\,+ & \dots & +\,101 \end{array}$$

So *double* the sum we want is equal to $100 \times 101 = 10100$, meaning our actual sum $1+2+\ldots+100$ equals 5050. The general formula for the nth triangular number is derived the same way (see the next problem)!

3. Where have we seen the above sequence of numbers $(1, 3, 6, 10, \ldots)$ before? Can you find (and prove!) a formula for the above sequence of numbers (that goes $1, 3, 6, 10, \ldots$?)

Solution: The triangular numbers pop up in the second diagonal of Pascal's Triangle (O_O). Here they are!



Using the same argument as in the solution of Problem 2, we see that $1+2+\ldots+n = \frac{n(n+1)}{2}$.
Hint!! Double the sum and line it up nicely like this:

$$\begin{array}{rrrrr} & 1\,+ & 2\,+ & \dots & +\,\text{n} \\ + & \text{n}\,+ & \text{(n-1)}\,+ & \dots & +\,1 \\ \hline & \text{(n+1)}\,+ & \text{(n+1)}\,+ & \dots & +\,\text{(n+1)} \end{array}$$

4. Say we decided to build a four dimensional Pascal's Triangle. How would you need to keep track of the numbers in a 4D Pascal's Triangle? Can you try to predict what those numbers would be using polynomials?

Solution: Just like every layer of a 3D Pascal's Triangle could be written down in the shape of a triangle, we would expect "every layer" of a 4D Pascal's Triangle to be in the shape of a tetrahedron (the shape that is one dimension lower).

We could predict the numbers in the nth "layer" by calculating the coefficients on all the different terms when we expand $(a+b+c+d)^n$. Alternatively, an easier formula might come up later in this Problem Set...

5. If we decided to count the number of circles in "layers" of tetrahedrons (hmm... this might be a hint for the previous question), what sequence of numbers would we get? What do you think the pattern would be?

Solution: Just like the triangular numbers, we'd get a sequence that we might call the **tetrahedral numbers**, which will count how many numbers there are in each "layer" of the 4D Pascal's Triangle. We can see these tetrahedral numbers pop up in Pascal's Triangle (see Problem 3 solution).

Bonus: We've seen that the n-triangular number is equal to $1+2+3+\ldots+n$. What do you think could be a formula for the n-tetrahedral number?

Formula-One

6. Last week, we might have seen the *closed form* formula

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

(If this looks unfamiliar, or you don't know how to prove this, check out the Week 1 Problem Set!). Using the same "Word Problem" interpretation as in Week 1, try to find a (similar) formula for the numbers in our 3D Pascal's Triangle!

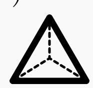
Note: I didn't give you a symbol to use as notation for these numbers[1], like I gave you the symbol $\binom{n}{k}$, so as a good first step you could come up with a symbol on your own! Remember, notation is just a *mathematician's shorthand*, so there's no right answer. You only need to find a shorthand that makes sense for you.

Bonus: Does your symbol generalize nicely to higher dimensions?

Solution: I'm going to come up with my own symbol! This could be very different from yours (and might not be the best symbol to use at all)! Here it is:

$$\binom{n}{k_1|k_2} = \text{the number of paths that are } n \text{ steps long, with } k_1 \text{ steps left and } k_2 \text{ steps right}$$

If we fix an orientation for our 3D Pascal's Triangle (for example, suppose it looks like this when we are looking at it from above):



Then there are three directions we could possibly choose from when we are moving "downwards" from the top: left, right, and forward (sometimes, it might be impossible to move in one of these directions, but that's alright).

So picking an intersection that is n steps away from the top, we can count and see that we'll need to move k_1 steps in one direction, k_2 steps in a second direction, and $n-k_1-k_2$ is the third remaining direction. For simplicity, let's call $k_3 = n-k_1-k_2$, so that $k_1+k_2+k_3 = n$. In other words, just as we saw in class, we are looking at the coefficient on $a^{k_1}b^{k_2}c^{k_3}$ in the trinomial expansion

$$(a+b+c)^n$$

In OTHER other words, we are looking at the number of words (remember we defined a **word** to be a sequence of letters in some order) we can make using the letters

$$\overbrace{aa\ldots}^{k_1}\overbrace{bb\ldots b}^{k_2}\overbrace{cc\ldots c}^{k_3}$$

Why? Because a particular word using those letters will tell us exactly how we multiplied

[1] There's a very good reason for this: there's no widespread symbol for trinomial expansion coefficients.

to get a term $a^{k_1}b^{k_2}c^{k_3}$. For example, if $n = 4$ and we are multiplying

$$(a+b+c)(a+b+c)(a+b+c)(a+b+c)$$

and say we are looking at the coefficient on a^2bc, then a word like $acba$ would tell us to pick the a from the first $(a + b + c)$, pick the c from the second $(a + b + c)$, pick the b from the third $(a + b + c)$, and pick the a from the fourth $(a + b + c)$. So how many total words are there using these letters? In alphabetical order, $aabc, aacb, abac, abca, acab, acba, baac, baca, bcaa, caab, caba, cbaa$. That's 12 words! And since each word "tells us" how to multiply to get a^2bc, the total coefficient on a^2bc in the expansion should be exactly 12 (the number of words).

PHEW, JUST TO RECAP: the number of the words using k_1 number of a's, k_2 number of b's, and k_3 number of c's is the number of ways to multiply to get $a^{k_1}b^{k_2}c^{k_3}$, which is exactly the *coefficient* on this term in the expansion of $(a + b + c)^n$. .

But we already figured out how to count the number of words like this, in last week's problem set![a] So

$$\binom{n}{k_1|k_2} = \frac{n!}{k_1!k_2!k_3!}$$

where the ! symbol (called a **factorial**) on a number means multiply the number by all the numbers less than it, up to 1.

Wow! The nice symmetry in the formula is so pleasant. Do you see why I defined k_3 the way I did? For a review of why I got to this formula so fast, check out the "Word Problems" in the Week 1 Problem Set!

And check back next week for more cool problems...

[a] Yay! That means less work for me ;)